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Viscosity approximation method for generalized asymptotically quasi-nonexpansive mappings in a convex metric space

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Abstract

A general viscosity iterative method for a finite family of generalized asymptotically quasi-nonexpansive mappings in a convex metric space is introduced. Special cases of the new iterative method are the viscosity iterative method of Chang *et al.* (Appl. Math. Comput. 212:51-59, 2009), an analogue of the viscosity iterative method of Fukhar-ud-din *et al.* (J. Nonlinear Convex Anal. 16:47-58, 2015) and an extension of the multistep iterative method of Yildirim and Özdemir (Arab. J. Sci. Eng. 36:393-403, 2011). Our results generalize and extend the corresponding known results in uniformly convex Banach spaces and CAT(0) spaces simultaneously.

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1 Introduction and preliminaries

Let C be a nonempty subset of a metric space X and $T : C \to C$ be a mapping. We assume that $F(T)$, the set of fixed points of T, is nonempty and $I = \{1, 2, 3, \dots, r\}$. The mapping T is (i) quasi-nonexpansive if $d(Tx, Ty) \le d(x, y)$ for $x \in C$, $y \in F(T)$; (ii) asymptotically quasi-nonexpansive if there exists a sequence of real numbers $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, p) \le (1 + u_n)d(x, p)$ for all $x \in C$, $p \in F(T)$ and $n \ge 1$; (iii) generalized asymptotically quasi-nonexpansive [1] if there exist two sequences of real numbers $\{u_n\}$ and $\{c_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0 = \lim_{n\to\infty} c_n$ such that $d(T^n x, p) \le d(x, p) + u_n d(x, p) + c_n$ for all $x \in C$, $p \in F(T)$ and $n \ge 1$; (iv) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \le Ld(x, y)$ for all $x, y \in C$ and $n \ge 1$; (v) uniformly Hölder continuous if there are constants $L > 0$, $\gamma > 0$ such that $d(T^n x, T^n y) \le Ld(x, y)^\gamma$ for all $x, y \in C$ and $n \ge 1$; and (vi) semi-compact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges to a point in C.

Clearly, the class of generalized asymptotically quasi-nonexpansive mappings includes the class of asymptotically quasi-nonexpansive mappings.

The following example improves and extends Example 3.2 in [1] to a finite family of generalized asymptotically quasi-nonexpansive mappings.

Example 1.1 Let $E = \mathbb{R}$ and $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and define $T_i x = \frac{x}{i+1}\sin(\frac{1}{x})$ if $x \neq 0$ and $T_i x = 0$ if $x = 0$ for all $x \in C$ and $i \in I$. Then $T_i^n x \to 0$ uniformly (see [2]). For each fixed n, define $f_{in}(x) = \|T_i^n x\| - \|x\|$ for all x in C and $i \in I$. Set $c_{in} = \sup_{x\in C}\{f_{in}(x), 0\}$. Then $\lim_{n\to\infty} c_{in} = 0$ and

$$\left\| T_i^n x\right\| \leq \|x\| + c_{in}.$$

This shows that $\{T_i : i \in I\}$ is a finite family of generalized asymptotically quasi-nonexpansive mappings with $\bigcap_{i\in I} F(T_i) \neq \emptyset$.

Convergence theorems for various mappings through different iterative methods have been obtained by a number of authors (*e.g.*, [1, 3, 4] and the references therein). For more on the study of fixed point iteration process, the interested reader is referred to Berinde [5] and Ciric [6, 7].

Let C be a convex subset of a normed space. Yildirim and Özdemir [8] introduced the following multistep iterative method:

$$\begin{aligned}
&x_1 \in C,\\
&x_{n+1} = (1 - a_{1n})y_{n+r-2} + a_{1n}T_1^n y_{n+r-2},\\
&y_{n+r-2} = (1 - a_{2n})y_{n+r-3} + a_{2n}T_2^n y_{n+r-3},\\
&\vdots\\
&y_{n+1} = (1 - a_{(r-1)n})y_n + a_{(r-1)n}T_{(r-1)}^n y_n,\\
&y_n = (1 - a_{rn})x_n + a_{rn}T_r^n x_n, \quad r \geq 2, n \geq 1,
\end{aligned} \tag{1.1}$$

where $\{T_i : i \in I\}$ is a family of self-mappings of C, $a_{in} \in [\epsilon, 1 - \epsilon]$, for some $\epsilon \in (0, \frac{1}{2})$, for all $n \geq 1$.

If $T_1 = T_2 = \cdots = T_r$ and $\alpha_{jn} = 0$ for $j = 1, \dots, r$ and $r \geq 1$, then the iterative method (1.1) reduces to the Mann iterative method [9]. Let us note that the scheme (1.1) and multistep scheme (1.3) in [10] are independent of each other.

Moudafi [11] proposed a viscosity iterative method by selecting a particular fixed point of a given nonexpansive mapping. The so-called viscosity iterative method has been studied by many authors (see, for example, [3, 12]). These methods are very important because of their applicability to convex optimization, linear programming, monotone inclusions and elliptic differential equations [11].

Recently, Chang *et al.* [13] introduced and studied the following viscosity iterative method:

$$\begin{aligned}
&x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n T^n y_n,\\
&y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1,
\end{aligned} \tag{1.2}$$

where T is an asymptotically nonexpansive mapping [14] and f is a fixed contraction.

The iterative methods in (1.1) and (1.2) involve convex combinations, and so a convex structure is needed to define them on a nonlinear domain.

A mapping $W : X^2 \times J \to X$ is a convex structure [15] on a metric space X if

$$d\big(u, W(x,y,\alpha)\big) \leq \alpha d(u,x) + (1-\alpha)d(u,y)$$

for all $x,y,u \in X$ and $\alpha \in J = [0,1]$. The metric space X together with a convex structure W is known as a convex metric space. A nonempty subset C of a convex metric space X is convex if $W(x,y,\alpha) \in C$ for all $x,y \in C$ and $\alpha \in J$. All normed linear spaces are convex metric spaces, but there are convex metric spaces which are not linear; for example, a CAT(0) space [16, 17].

A convex metric space X is uniformly convex if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $r > 0$ and $x,y,z \in X$ with $d(z,x) \leq r$, $d(z,y) \leq r$ and $d(x,y) \geq r\varepsilon$ imply that $d(z, W(x,y,\frac{1}{2})) \leq (1-\delta)r$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ϵ).

Obviously, uniformly convex Banach spaces are uniformly convex metric spaces.

In general, a convex structure W is not continuous [18]. Throughout this paper, we assume that W is continuous.

We now devise a general iterative method which extends the methods in (1.1) and (1.2) simultaneously in a convex metric space.

We define an S_n-mapping generated by a family $\{T_i : i \in I\}$ of generalized asymptotically quasi-nonexpansive mappings on C as

$$S_n x = U_{rn} x, \tag{1.3}$$

where $U_{0n} = I$ (the identity mapping), $U_{1n}x = W(T_r^n U_{0n}x, U_{0n}x, a_{rn})$, $U_{2n}x = W(T_{r-1}^n U_{1n}x, U_{1n}x, a_{(r-1)n}), \ldots, U_{rn}x = W(T_1^n U_{(r-1)n}x, U_{(r-1)n}x, a_{1n})$.

For $\{\alpha_n\} \subset J$, a fixed contractive mapping f on C and S_n given in (1.3), we define $\{x_n\}$ as follows:

$$x_1 \in C, \quad x_{n+1} = W\big(f(x_n), S_n x_n, \alpha_n\big) \tag{1.4}$$

and call it a general viscosity iterative method in a convex metric space.

The purpose of this paper is to:

(i) establish a necessary and sufficient condition for convergence of iterative method (1.4) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings on a convex metric space;

(ii) prove strong convergence and $\triangle$-convergence results for the iterative method (1.4) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings on a uniformly convex metric space.

We now assume that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$.

We need the following known results for our convergence analysis.

Lemma 1.1 (*cf.* [19]) *Let the sequences* $\{a_n\}$ *and* $\{u_n\}$ *of real numbers satisfy*

$$a_{n+1} \leq (1+u_n)a_n, \quad a_n \geq 0, u_n \geq 0, \sum_{n=1}^{\infty} u_n < +\infty.$$

Then (i) $\lim_{n\to\infty} a_n$ *exists*; (ii) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2 ([20]) *Let* X *be a uniformly convex metric space. Let* $x \in X$ *and* $\{a_n\}$ *be a sequence in* $[b,c]$ *for some* $b,c \in (0,1)$. *If* $\{u_n\}$ *and* $\{v_n\}$ *are sequences in* X *such that* $\limsup_{n\to\infty} d(u_n,x) \leq r$, $\limsup_{n\to\infty} d(v_n,x) \leq r$ *and* $\lim_{n\to\infty} d(W(u_n,v_n,a_n),x) = r$ *for some* $r \geq 0$, *then* $\lim_{n\to\infty} d(u_n,v_n) = 0$.

2 Convergence in convex metric spaces

In this section, we prove some results for the viscosity iterative method (1.4) to converge to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in a convex metric space.

Lemma 2.1 *Let* C *be a nonempty, closed and convex subset of a convex metric space* X *and* $\{T_i : i \in I\}$ *be a family of generalized asymptotically quasi-nonexpansive self-mappings of* C, *i.e.*, $d(T_i^n x, p_i) \leq (1+u_{in})d(x,p_i) + c_{in}$ *for all* $x \in C$ *and* $p_i \in F(T_i)$, $i \in I$, *where* $\{u_{in}\}$ *and* $\{c_{in}\}$ *are sequences in* $[0,\infty)$ *with* $\sum_{n=1}^{\infty} u_{in} < \infty$, $\sum_{n=1}^{\infty} c_{in} < \infty$ *for each* i. *Then, for the sequence* $\{x_n\}$ *in* (1.4) *with* $\sum_{n=1}^{\infty} \alpha_n < \infty$, *there are sequences* $\{v_n\}$ *and* $\{\xi_n\}$ *in* $[0,\infty)$ *satisfying* $\sum_{n=1}^{\infty} v_n < \infty$, $\sum_{n=1}^{\infty} \xi_n < \infty$ *such that*

(a) $d(x_{n+1},p) \leq (1+v_n)^r d(x_n,p) + \xi_n$ *for all* $p \in F$ *and all* $n \geq 1$;

(b) $d(x_{n+m},p) \leq M_1(d(x_n,p) + \sum_{n=1}^{\infty} \xi_n)$ *for all* $p \in F$ *and* $n \geq 1$, $m \geq 1$, $M_1 > 0$.

Proof (a) Let $p \in F$ and $v_n = \max_{i\in I} u_{in}$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} u_{in} < \infty$ for each i, therefore $\sum_{n=1}^{\infty} v_n < \infty$.

Now we have

$$\begin{aligned} d(U_{1n}x_n,p) &= d\big(W\big(T_r^n U_{0n}x_n, U_{0n}x_n, a_{rn}\big),p\big) \\ &\leq (1-a_{rn})d(x_n,p) + a_{rn}d\big(T_r^n x_n,p\big) \\ &\leq (1-a_{rn})d(x_n,p) + a_{rn}\big[(1+u_{rn})d(x_n,p) + c_{rn}\big] \\ &\leq (1+u_{rn})d(x_n,p) + c_{rn} \\ &\leq (1+v_n)^1 d(x_n,p) + c_{rn}. \end{aligned}$$

Assume that $d(U_{kn}x_n,p) \leq (1+v_n)^k d(x_n,p) + (1+v_n)^{k-1}\sum_{i=1}^{k} c_{(r-i+1)n}$ holds for some $1 < k$.

Consider

$$\begin{aligned} d(U_{(k+1)n}x_n,p) &= d\big(W\big(T_{r-k}^n U_{kn}x_n, U_{kn}x_n, a_{(r-k)n}\big),p\big) \\ &\leq (1-a_{(r-k)n})d(U_{kn}x_n,p) + a_{(r-k)n}d\big(T_{r-k}^n U_{kn}x_n,p\big) \\ &\leq (1-a_{(r-k)n})d(U_{kn}x_n,p) + a_{(r-k)n}\big[(1+u_{(r-k)n})d(U_{kn}x_n,p) + c_{(r-k)n}\big] \\ &\leq (1+v_n)d(U_{kn}x_n,p) + c_{(r-k)n} \end{aligned}$$

$$\leq (1+\nu_n)\left[(1+\nu_n)^k d(x_n,p) + (1+\nu_n)^{k-1}\sum_{i=1}^{k} c_{(r-i+1)n}\right] + c_{(r-k)n}$$

$$\leq (1+\nu_n)^{k+1} d(x_n,p) + (1+\nu_n)^{k}\sum_{i=1}^{k+1} c_{(r-i+1)n}.$$

By mathematical induction, we have

$$d(U_{jn}x_n,p) \leq (1+\nu_n)^j d(x_n,p) + (1+\nu_n)^{j-1}\sum_{i=1}^{j} c_{(r-i+1)n}, \quad 1\leq j\leq r. \tag{2.1}$$

Hence

$$d(S_nx_n,p) = d(U_{rn}x_n,p) \leq (1+\nu_n)^r d(x_n,p) + (1+\nu_n)^{r-1}\sum_{i=1}^{r} c_{(r-i+1)n}. \tag{2.2}$$

Now, by (1.4) and (2.2), we obtain

$$\begin{aligned} d(x_{n+1},p) &= d\bigl(W\bigl(f(x_n),S_nx_n,\alpha_n\bigr),p\bigr) \\ &\leq \alpha_n d\bigl(f(x_n),p\bigr) + (1-\alpha_n)d(S_nx_n,p) \\ &\leq \alpha_n d(x_n,p) + \alpha_n d\bigl(f(p),p\bigr) \\ &\quad + (1-\alpha_n)\left((1+\nu_n)^r d(x_n,p) + (1+\nu_n)^{r-1}\sum_{i=1}^{r} c_{(r-i+1)n}\right) \\ &\leq (1+\nu_n)^r d(x_n,p) + (1-\alpha_n)(1+\nu_n)^{r-1}\sum_{i=1}^{r} c_{(r-i+1)n} + \alpha_n d\bigl(f(p),p\bigr) \\ &\leq (1+\nu_n)^r d(x_n,p) + \alpha_n d\bigl(f(p),p\bigr) + (1+\nu_n)^{r-1}\sum_{i=1}^{r} c_{(r-i+1)n}. \end{aligned}$$

Setting $\max\{d(f(p),p), \sup(1+\nu_n)^{r-1}\} = M$, we get that

$$d(x_{n+1},p) \leq (1+\nu_n)^r d(x_n,p) + M\left(\alpha_n + \sum_{i=1}^{r} c_{(r-i+1)n}\right).$$

That is,

$$d(x_{n+1},p) \leq (1+\nu_n)^r d(x_n,p) + \xi_n,$$

where $\xi_n = M(\alpha_n + \sum_{i=1}^{r} c_{(r-i+1)n})$ and $\sum_{n=1}^{\infty}\xi_n < \infty$.

(b) We know that $1+t\leq e^t$ for $t\geq 0$. Thus, by part (a), we have

$$\begin{aligned} d(x_{n+m},p) &\leq (1+\nu_{n+m-1})^r d(x_{n+m-1},p) + \xi_{n+m-1} \\ &\leq e^{r\nu_{n+m-1}} d(x_{n+m-1},p) + \xi_{n+m-1} \\ &\leq e^{r(\nu_{n+m-1}+\nu_{n+m-2})} d(x_{n+m-2},p) + \xi_{n+m-1} + \xi_{n+m-2} \\ &\vdots \end{aligned}$$

$$\begin{aligned}
&\le e^{r\sum_{i=n}^{n+m-1}\nu_i} d(x_n,p) + \sum_{i=n+1}^{n+m-1}\nu_i \sum_{i=n}^{n+m-1}\xi_i \\
&\le e^{r\sum_{i=1}^{\infty}\nu_i}\left(d(x_n,p) + \sum_{i=1}^{\infty}\xi_i\right) \\
&= M_1\left(d(x_n,p) + \sum_{i=1}^{\infty}\xi_i\right), \quad \text{where } M_1 = e^{r\sum_{i=1}^{\infty}\nu_i}.
\end{aligned}$$

□

The next result deals with a necessary and sufficient condition for the convergence of $\{x_n\}$ in (1.4) to a point of F.

Theorem 2.1 *Let C, $\{T_i : i \in I\}$, F, $\{u_{in}\}$ and $\{c_{in}\}$ be as in Lemma* 2.1. *Let X be complete. The sequence $\{x_n\}$ in* (1.4) *with $\sum_{n=1}^{\infty}\alpha_n < \infty$ converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$, where $d(x,F) = \inf_{p\in F}(x,p)$.*

Proof The necessity is obvious; we only prove the sufficiency. By Lemma 2.1(a), we have

$$d(x_{n+1},p) \le (1+\nu_n)^r d(x_n,p) + \xi_n \quad \text{for all } p \in F \text{ and } n \ge 1.$$

Therefore,

$$\begin{aligned}
d(x_{n+1},F) &\le (1+\nu_n)^r d(x_n,F) + \xi_n \\
&= \left(1 + \sum_{k=1}^{r}\frac{r(r-1)\cdots(r-k+1)}{k!}\nu_n^k\right) d(x_n,F) + \xi_n.
\end{aligned}$$

As $\sum_{n=1}^{\infty}\nu_n < +\infty$, so $\sum_{n=1}^{\infty}\sum_{k=1}^{r}\frac{r(r-1)\cdots(r-k+1)}{k!}\nu_n^k < \infty$. Now $\sum_{n=1}^{\infty}\xi_n < \infty$ in Lemma 2.1(a), so by Lemma 1.1 and $\liminf_{n\to\infty} d(x_n,F) = 0$, we get that $\lim_{n\to\infty} d(x_n,F) = 0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence in X. Let $\varepsilon > 0$. From the proof of Lemma 2.1(b), we have

$$d(x_{n+m},x_n) \le d(x_{n+m},F) + d(x_n,F) \le (1+M_1)d(x_n,F) + M_1\sum_{i=n}^{\infty}\xi_i. \tag{2.3}$$

As $\lim_{n\to\infty} d(x_n,F) = 0$ and $\sum_{i=1}^{\infty}\xi_i < \infty$, so there exists a natural number n_0 such that

$$d(x_n,F) \le \frac{\varepsilon}{2(1+M_1)} \quad \text{and} \quad \sum_{i=n}^{\infty}\xi_i < \frac{\varepsilon}{2M_1} \quad \text{for all } n \ge n_0.$$

So, for all integers $n \ge n_0$, $m \ge 1$, we obtain from (2.3) that

$$d(x_{n+m},x_n) < (M_1+1)\frac{\varepsilon}{2(1+M_1)} + M_1\frac{\varepsilon}{2M_1} = \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X and so it converges to $q \in X$. Finally, we show that $q \in F$. For any $\overline{\varepsilon} > 0$, there exists a natural number n_1 such that

$$d(x_n,F) = \inf_{p\in F} d(x_n,p) < \frac{\overline{\varepsilon}}{3} \quad \text{and} \quad d(x_n,q) < \frac{\overline{\varepsilon}}{2} \quad \text{for all } n \ge n_1.$$

There must exist $p^* \in F$ such that $d(x_n, p^*) < \frac{\bar{\varepsilon}}{2}$ for all $n \geq n_1$; in particular, $d(x_{n_1}, p^*) < \frac{\bar{\varepsilon}}{2}$ and $d(x_{n_1}, q) < \frac{\bar{\varepsilon}}{2}$.

Hence

$$d(p^*, q) \leq d(x_{n_1}, p^*) + d(x_{n_1}, q) < \bar{\varepsilon}.$$

Since $\bar{\varepsilon}$ is arbitrary, therefore $d(p^*, q) = 0$. That is, $q = p^* \in F$. □

Remark 2.1 A generalized asymptotically nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping. So Theorem 2.1 holds good for the class of generalized asymptotically nonexpansive mappings.

3 Results in a uniformly convex metric space

The aim of this section is to establish some convergence results for the iterative method (1.4) of generalized asymptotically quasi-nonexpansive mappings on a uniformly convex metric space.

Lemma 3.1 *Let C be a nonempty, closed and convex subset of a uniformly convex metric space X and $\{T_i : i \in I\}$ be a family of uniformly Hölder continuous and generalized asymptotically quasi-nonexpansive self-mappings of C, i.e., $d(T_i^n x, p_i) \leq (1 + u_{in})d(x, p_i) + c_{in}$ for all $x \in C$ and $p_i \in F(T_i)$, where $\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$, respectively, for each $i \in I$. Then, for the sequence $\{x_n\}$ in (1.4) with $a_{in} \in [\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$, we have $\lim_{n\to\infty} d(x_n, T_j x_n) = 0$ for each $j \in I$.*

Proof Let $p \in F$ and $v_n = \max_{i\in I} u_{in}$ for all $n \geq 1$. By Lemma 1.1(i) and Lemma 2.1(a), it follows that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. Assume that

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{3.1}$$

Inequality (2.1) together with (3.1) gives that

$$\limsup_{n\to\infty} d(U_{jn} x_n, p) \leq c, \quad 1 \leq j \leq r. \tag{3.2}$$

By (1.4), we have

$$\begin{aligned} d(x_{n+1}, p) &= d\big(W\big(f(x_n), S_n x_n, \alpha_n\big), p\big) \\ &\leq \alpha_n d\big(f(x_n), p\big) + (1 - \alpha_n) d(S_n x_n, p) \\ &\leq \alpha_n d\big(f(x_n), p\big) + \alpha_n d\big(f(p), p\big) + (1 - \alpha_n) d(U_{rn} x_n, p), \end{aligned}$$

and hence

$$c \leq \liminf_{n\to\infty} d(U_{rn} x_n, p). \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\lim_{n\to\infty} d(U_{rn} x_n, p) = c.$$

Note that

$$\begin{aligned}
d(U_{rn}x_n,p) &= d\big(W\big(T_1^n U_{(r-1)n}x_n, U_{(r-1)n}x_n, a_{1n}\big),p\big) \\
&\le a_{1n}d\big(T_1^n U_{(r-1)n}x_n,p\big) + (1-a_{1n})d(U_{(r-1)n}x_n,p) \\
&\le a_{1n}\big[(1+u_{1n})d(U_{(r-1)n}x_n,p) + c_{1n}\big] + (1-a_{1n})d(U_{(r-1)n}x_n,p) \\
&\le a_{1n}(1+v_n)d(U_{(r-1)n}x_n,p) + a_{1n}c_{1n} \\
&\le a_{1n}(1+v_n)\big[a_{2n}(1+v_n)d(U_{(r-2)n}x_n,p) + a_{2n}c_{2n}\big] + a_{1n}(1+v_n)c_{1n} \\
&\le a_{1n}a_{2n}(1+v_n)^2 d(U_{(r-2)n}x_n,p) + a_{1n}a_{2n}(1+v_n)c_{2n} + a_{1n}c_{1n} \\
&\quad \vdots \\
&\le a_{1n}a_{2n}\cdots a_{(j-1)n}(1+v_n)^{j-1}d(U_{(r-(j-1))n}x_n,p) \\
&\quad + a_{1n}a_{2n}\cdots a_{(j-1)n}(1+v_n)^{(j-1)-1}c_{(j-1)n} \\
&\quad + a_{1n}a_{2n}\cdots a_{((j-1)-1)n}(1+v_n)^{(j-1)-2}c_{((j-1)-1)n} + \cdots \\
&\quad + a_{1n}a_{2n}(1+v_n)c_{2n} + a_{1n}c_{1n}.
\end{aligned}$$

Hence

$$c \le \liminf_{n\to\infty} d(U_{(r-(j-1))n}x_n,p), \quad 1 \le j \le r. \tag{3.4}$$

Using (3.2) and (3.4), we have

$$\lim_{n\to\infty} d(U_{(r-(j-1))n}x_n,p) = c.$$

That is,

$$\lim_{n\to\infty} d\big(W\big(T_j^n U_{(r-j)n}x_n, U_{(r-j)n}x_n, a_{jn}\big),p\big) = c \quad \text{for } 1 \le j \le r.$$

This together with (3.1), (3.2) and Lemma 1.2 gives that

$$\lim_{n\to\infty} d\big(T_j^n U_{(r-j)n}x_n, U_{(r-j)n}x_n,\big) = 0 \quad \text{for } 1 \le j \le r. \tag{3.5}$$

If $j = r$,we have by (3.5)

$$\lim_{n\to\infty} d\big(T_r^n x_n, x_n\big) = 0.$$

In case $j \in \{1,2,3,\ldots,r-1\}$, we observe that

$$\begin{aligned}
d(x_n, U_{(r-j)n}x_n) &= d\big(x_n, W\big(T_{j+1}^n U_{(r-(j+1))n}x_n, U_{(r-(j+1))n}x_n, a_{(j+1)n}\big)\big) \\
&\le a_{(j+1)n}d\big(T_{j+1}^n U_{(r-(j+1))n}x_n, x_n\big) + (1-a_{(j+1)n})d(U_{(r-(j+1))n}x_n,x_n) \\
&\le (1+v_n)d(U_{(r-(j+1))n}x_n,x_n) + c_{(j+1)n} \\
&\quad \vdots
\end{aligned}$$

$$\begin{aligned}&\le (1+\nu_n)^{r-j} d(U_{0n}x_n, x_n) + (1+\nu_n)^{r-j-1} c_{rn}\\&\quad + (1+\nu_n)^{r-j-2} c_{(r-1)n} + \cdots + (1+\nu_n) c_{(j+2)n} + c_{(j+1)n}.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} d(x_n, U_{(r-j)n}x_n) = 0. \tag{3.6}$$

Since T_j is uniformly Hölder continuous, therefore the inequality

$$\begin{aligned}d\bigl(T_j^n x_n, x_n\bigr) &\le d\bigl(T_j^n x_n, T_j^n U_{(r-j)n}x_n\bigr) + d\bigl(T_j^n U_{(r-j)n}x_n, U_{(r-j)n}x_n\bigr)\\&\quad + d(U_{(r-j)n}x_n, x_n)\\&\le L d(x_n, U_{(r-j)n}x_n)^{\gamma} + d(x_n, U_{(r-j)n}x_n) + d\bigl(T_j^n U_{(r-j)n}x_n, U_{(r-j)n}x_n\bigr)\end{aligned}$$

together with (3.5) and (3.6) gives that

$$\lim_{n\to\infty} d\bigl(T_j^n x_n, x_n\bigr) = 0.$$

Hence,

$$d\bigl(T_j^n x_n, x_n\bigr) \to 0 \quad \text{as } n\to\infty \text{ for } 1\le j\le r. \tag{3.7}$$

As before, we can show that

$$\begin{aligned}d(x_n, x_{n+1}) &= d\bigl(x_n, W\bigl(f(x_n), S_n x_n, \alpha_n\bigr)\bigr)\\&\le \alpha_n(1+\alpha) d(x_n, p) + \alpha_n d\bigl(p, f(p)\bigr)\\&\quad + (1-\alpha_n)\bigl[a_{1n} d\bigl(U_{(r-1)n}x_n, T_1^n U_{(r-1)n}x_n\bigr) + d(x_n, U_{(r-1)n}x_n)\bigr].\end{aligned}$$

Therefore, by (3.5) and (3.6), we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

Let us observe that

$$\begin{aligned}d(x_n, T_j x_n) &\le d(x_n, x_{n+1}) + d\bigl(x_{n+1}, T_j^{n+1} x_{n+1}\bigr)\\&\quad + d\bigl(T_j^{n+1} x_{n+1}, T_j^{n+1} x_n\bigr) + d\bigl(T_j^{n+1} x_n, T_j x_n\bigr)\\&\le d(x_n, x_{n+1}) + d\bigl(x_{n+1}, T_j^{n+1} x_{n+1}\bigr)\\&\quad + L d(x_{n+1}, x_n)^{\gamma} + L d\bigl(T_j^n x_n, x_n\bigr)^{\gamma}.\end{aligned}$$

By the uniform Hölder continuity of T_j, (3.7) and (3.8), we get

$$\lim_{n\to\infty} d(x_n, T_j x_n) = 0, \quad 1\le j\le r. \tag{3.9}$$

□

Theorem 3.1 *Under the hypotheses of Lemma* 3.1, *assume, for some* $1 \leq j \leq r$, *that* T_j^m *is semi-compact for some positive integer* m. *If* X *is complete, then* $\{x_n\}$ *in* (1.4) *converges strongly to a point in* F.

Proof Fix $j \in I$ and suppose T_j^m to be semi-compact for some $m \geq 1$. By (3.9), we obtain

$$\begin{aligned} d\bigl(T_j^m x_n, x_n\bigr) &\leq d\bigl(T_j^m x_n, T_j^{m-1} x_n\bigr) + d\bigl(T_j^{m-1} x_n, T_j^{m-2} x_n\bigr) \\ &\quad + \cdots + d\bigl(T_j^2 x_n, T_j x_n\bigr) + d(T_j x_n, x_n) \\ &\leq d(T_j x_n, x_n) + (m-1) L d(T_j x_n, x_n)^\gamma \to 0. \end{aligned}$$

Since $\{x_n\}$ is bounded and T_j^m is semi-compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to q \in C$. Hence, by (3.9), we have

$$d(q, T_i q) = \lim_{n\to\infty} d(x_{n_j}, T_i x_{n_j}) = 0, \quad i \in I.$$

Thus $q \in F$, and so by Theorem 2.1, $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i \in I\}$. □

An immediate consequence of Lemma 3.1 and Theorem 3.1 is the following strong convergence result in uniformly convex metric spaces.

Theorem 3.2 *Let* C, $\{T_i : i \in I\}$, F, $\{u_{in}\}$ *and* $\{c_{in}\}$ *be as in Lemma* 3.1. *If there exists a constant* M *such that* $d(x_n, T_j x_n) \geq M d(x_n, F)$ *for all* $n \geq 1$ *and* X *is complete, then the sequence* $\{x_n\}$ *in* (1.4) *converges strongly to a point in* F.

The concept of $\triangle$-convergence in a metric space was introduced by Lim [21] and its analogue in CAT(0) spaces was investigated by Dhompongsa and Panyanak [22]. Here we study $\triangle$-convergence in uniformly convex metric spaces.

For this, we collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a uniformly convex metric space X. For $x \in X$, define a continuous functional $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by

$$\rho = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in X\bigr\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset C of X is defined as follows:

$$A_C\bigl(\{x_n\}\bigr) = \bigl\{x \in X : r\bigl(x, \{x_n\}\bigr) \leq r\bigl(y, \{x_n\}\bigr) \text{ for any } y \in C\bigr\}.$$

If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. A sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_n x_n = x$ and call x as $\triangle$-limit of $\{x_n\}$.

Lemma 3.2 ([23]) *Let (X,d) be a complete uniformly convex metric space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

Lemma 3.3 ([20]) *Let C be a nonempty closed convex subset of a uniformly convex metric space and $\{x_n\}$ be a bounded sequence in C such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Now, we establish $\triangle$-convergence of the iterative method (1.4).

Theorem 3.3 *Let C be a nonempty, closed and convex subset of a complete uniformly convex metric space X with monotone modulus of uniform convexity η, and let $\{T_i : i \in I\}$ be a family of uniformly L-Lipschitzian and generalized asymptotically nonexpansive self-mappings of C such that $F \neq \phi$, i.e., $d(T_i^n x, T_i^n y) \leq (1 + u_{in})d(x,y) + c_{in}$ for all $x, y \in C$, where $\{u_{in}\}$ and $\{c_{in}\}$ are sequences in $[0,\infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$, respectively, for each $i \in I$. Then the sequence $\{x_n\}$ in (1.4) with $a_{in} \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\triangle$-converges to a common fixed point of $\{T_j : j \in I\}$.*

Proof By Lemma 3.1, $\{x_n\}$ is bounded, and so by Lemma 3.2, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Let $\{z_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{z_n\}) = \{z\}$. Also by Lemma 3.1, we have $\lim_{n\to\infty} d(z_n, T_j z_n) = 0$ for each $j \in I$.

We claim that z is a common fixed point of $\{T_j : j \in I\}$. To show this, we define a sequence $\{w_k\}$ in C by $w_k = T_j^k z$,

$$\begin{aligned} d(w_k, z_n) &= d\bigl(T_j^k z, z_n\bigr) \\ &\leq d\bigl(T_j^k z, T_j^k z_n\bigr) + \sum_{i=1}^{k} d\bigl(T_j^i z_n, T_j^{i-1} z_n\bigr) \\ &\leq (1 + u_{jn})d(z, z_n) + c_{jn} + kLd(T_j z_n, z_n). \end{aligned}$$

Taking lim sup,

$$\limsup_{n\to\infty} d(w_k, z_n) \leq \limsup_{n\to\infty} d(z, z_n),$$

i.e., $r(T_j^k z, z_n) \leq r(z, z_n)$. It follows from Lemma 3.3 that $\lim_{k\to\infty} T_j^k z = z$. As T_j is uniformly continuous, we have $T_j z = T_j(\lim_{k\to\infty} T_j^k z) = \lim_{k\to\infty} T_j^{k+1} z = z$. Therefore, z is a common fixed point of $\{T_j : j \in I\}$.

Recall that $\lim_{n\to\infty} d(x_n, z)$ exists by Lemma 3.1.

Suppose $x \neq z$. By the uniqueness of asymptotic centers, we obtain

$$\begin{aligned} \limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, z) \\ &= \limsup_{n\to\infty} d(z_n, z), \end{aligned}$$

a contradiction. Hence $x = z$. Since $\{z_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{z_n\}) = \{z\}$ for all subsequences $\{z_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle$-converges to a common fixed point of $\{T_j : j \in I\}$. □

Remark 3.1

(i) Lemma 3.1, Theorems 3.1 and 3.3 set an analogue of Theorems 2.8-2.10 in [24] and Lemma 3.2, Theorems 3.4 and 3.5 in [25], in uniformly convex metric spaces.
(ii) Lemma 3.1 and Theorem 3.1 provide an analogue of Lemma 3.7 and Theorem 3.8 in [1] and Lemma 2.6 and Theorem 2.7 in [4] in uniformly convex metric spaces.
(iii) Theorems 2.1 and 3.3 extend Theorems 3.2, 3.6, and 3.7 in [8], to convex metric spaces.
(iv) Our results give an analogue of the results in [26].

Open problem Assume that the initial point is the same in scheme (1.1) and multistep scheme (1.3) in [10]. Under what conditions are these schemes equivalent?

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed to this work on an equal basis. All authors read and approved the final manuscript.

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